Malaysian Journal of Mathematical Sciences 12(2): 211–222 (2018)

MALAYSIAN JOURNAL OF MATHEMATICAL SCIENCES

Journal homepage: http://einspem.upm.edu.my/journal

Classification of Finite Simple Groups According to the Number of Centralizers

Foruzanfar, Z.[1], Mostaghim, Z. *[2], and Rezaei, M.[1]

[1] *Department of Mathematics, Buein Zahra Technical University, Buein Zahra, Qazvin, Iran*

[2] *School of Mathematics, Iran University of Science and Technology, Tehran, Iran*

E-mail: mostaghim@iust.ac.ir

* *Corresponding author*

Received: 2 November 2016

Accepted: 21 January 2018

ABSTRACT

Let G be a finite group and $|Cent(G)|$ be the number of distinct centralizers of its elements. G is called *n-centralizer* if $|Cent(G)| = n$. In this paper, we classify all finite non-abelian simple groups G with $|Cent(G)| \leq 100$.

Keywords: Finite group, n-centralizer group, simple group.

1. Introduction

Let G be a finite group. We denote by $Cent(G) := \{C_G(g)|g \in G\}$, where $C_G(g)$ is the centralizer of the element g in G. Let $n > 0$ be an integer. A group G is called *n-centralizer* if $|Cent(G)| = n$.

It is clear that a group is 1-centralizer if and only if it is abelian. Belcastro and Sherman proved the following results in Belcastro and Sherman (1994):

(i) There is no 2-centralizer and no 3-centralizer group.

(ii) A finite group G is 4-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(iii) A finite group G is 5-centralizer if and only if $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ or S_3.

Subsequently, all finite n-centralizer groups for $n \leq 9$ have been characterized in Abdollahi et al. (2007), Ashrafi (2000b), Ashrafi (2000a) and Foruzanfar and Mostaghim (2015). In Ashrafi and Taeri (2005), the structure of finite groups G with $|Cent(G)| \leq 21$ has been investigated and also using the classification of finite simple groups, the authors proved that if G is a finite simple group and $|Cent(G)| = 22$, then $G \cong A_5$. In Zarrin (2009), all finite semi-simple groups G with $|Cent(G)| \leq 73$ have been classified.

In this paper, we classify all finite non-abelian simple groups G with $|Cent(G)| \leq 100$ and prove the following theorem:

Theorem 1.1. *If G is a finite non-abelian simple group with $|Cent(G)| \leq 100$, then it is isomorphic to one of the following groups:*
$PSL_2(5), PSL_2(7)$ *or* $PSL_2(8)$.

2. Preliminary results

Let G be a finite group and p a prime divisor of the order of G. We denote by $\upsilon_p(G)$, the number of Sylow p-subgroups of G which pairwise intersect trivially. Also q denotes the order of a finite field and so is a prime power.

We first describe the classification theorem of finite simple groups and then present some lemmas that will be used in the proof of Theorem 1.1.

Theorem 2.1 (Wilson (2009), p. 3). *Every finite simple group is isomorphic to one of the following:*
(i) a cyclic group $\mathbb{Z}_p$ of prime order p;

(ii) an alternating group A_n *for* $n \geq 5$*;*
(iii) a classical group:

linear: $PSL_n(q)$ *for* $n \geq 2$*, except* $PSL_2(2)$ *and* $PSL_2(3)$*;*

unitary: $PSU_n(q)$ *for* $n \geq 3$*, except* $PSU_3(2)$*;*

symplectic: $PSp_{2n}(q)$ *for* $n \geq 2$*, except* $PSp_4(2)$*;*

orthogonal: $P\Omega_{2n+1}(q)$ *for* $n \geq 3$ *and* q *odd;*

$P\Omega^{+}_{2n}(q)$ *for* $n \geq 4$*;*

$P\Omega^{-}_{2n}(q)$ *for* $n \geq 4$*;*

(iv) an exceptional group of Lie type:

$$G_2(q) \text{ for } q \geq 3,\ F_4(q),\ E_6(q),\ {}^2E_6(q),\ {}^3D_4(q),\ E_7(q),\ E_8(q);$$

or

$${}^2B_2(2^{2m+1}) \cong Sz(2^{2m+1}) \text{ for } m \geq 1 \text{ (the Suzuki group);}$$

$${}^2F_4(2^{2m+1}),\ {}^2G_2(3^{2m+1}) \text{ for } m \geq 1 \text{ (the Ree groups);}$$

or the Tits group ${}^2F_4(2)'$.

(v) one of 26 sporadic simple groups:

the five Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}*;*

the seven Leech lattice groups Co_1, Co_2, Co_3, McL, HS, Suz, J_2*;*

the three Fischer groups Fi_{22}, Fi_{23}, Fi'_{24}*;*

the five Monstrous groups M, B, Th, HN, He*;*

the six pariahs J_1, J_3, J_4, $O'N$, Ly, Ru.

Conversely, every group in this list is simple, and the only repetitions in this list are:

$PSL_2(4) \cong PSL_2(5) \cong A_5$;

$PSL_2(7) \cong PSL_3(2)$;

$PSL_2(9) \cong A_6$;

$PSL_4(2) \cong A_8$;

$PSU_4(2) \cong PSp_4(3)$.

Lemma 2.1 (Ashrafi and Taeri (2005), Lemma 4). *Let G be a finite group, and p a prime divisor of the order of G. Then* $|Cent(G)| \geq \upsilon_p(G) + 1$.

Lemma 2.2 (Ashrafi and Taeri (2005), Lemma 5). *Let G be a finite group, and p a prime divisor of the order of G. If $p^2 \nmid |G|$, and G have more than one Sylow p-subgroup, then $|Cent(G)| \geqslant 2 + kp$, where k is the least positive integer such that $1 + kp$ divides $|G|$.*

Lemma 2.3 (Ashrafi and Taeri (2005), Lemma 6). *Let K be a subgroup of a finite group G. Then $|Cent(K)| \leqslant |Cent(G)|$.*

Theorem 2.2 (Zarrin (2009), Theorem 1.1). *Let $G = PSL_2(q)$, where q is a p-power (p prime). Then*

1. *If $q \in \{2, 3, 5\}$ or $q \equiv 0 \mod 4$, then*

$$|Cent(G)| = \left\{ \begin{array}{lll} q^2 + q + 2 & if & q > 5, \\ 22 & if & q = 4 \ \ or \ \ 5, \\ 6 & if & q = 3, \\ 5 & if & q = 2. \end{array} \right\}.$$

2. *If $q > 5$ and $q \equiv 1 \mod 4$, then*

$$|Cent(G)| = \frac{3q^2 + 3q + 4}{2}.$$

3. *If $q > 5$ and $q \equiv 3 \mod 4$, then*

$$|Cent(G)| = \frac{3q^2 + q + 4}{2}.$$

Theorem 2.3 (Zarrin (2009), Theorem 1.2). *Let $G = Sz(q)$ ($q = 2^{2m+1}, m > 0$). Then*

$$|Cent(G)| = q^3 - q^2 + q + \frac{q^2(q^2+1)}{2} + \frac{q^2(q^2+1)(q-1)}{4(q+2r+1)} + \frac{q^2(q^2+1)(q-1)}{4(q-2r+1)},$$

where $r = \sqrt{\frac{q}{2}}$.

3. The proof of Theorem 1.1

To prove our main result, Theorem 1.1, we first show the following lemmas.

Lemma 3.1. *If $G = PSL(n,q)$, where $(n,q) \neq (2,2),(2,3)$, then $|Cent(G)| > 100$, unless in the cases: $PSL_2(5), PSL_2(7)$ or $PSL_2(8)$.*

Proof. Let $q = p^m$, where p is prime and $m \geq 1$. Suppose that $|Cent(PSL_n(q))| \leq 100$. If $n = 2$, then by [Huppert (1967), Theorem 8.2, p. 191] we have

$$100 \geq |Cent(PSL_2(p^m))| \geq \upsilon_p(PSL_2(p^m)) + 1 = p^m + 2.$$

Thus we have the following cases: $p^m = 4, 8, 16, 32, 64, 9, 27, 81, 5, 25, 7, 49, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97$.

So by Theorem 2.2 only the following three groups will be remained:

$PSL_2(4) \cong PSL_2(5) \cong A_5, PSL_2(7)$ and $PSL_2(8)$

and we have

$|Cent(PSL_2(5))| = 22, |Cent(PSL_2(7))| = 79$ and $|Cent(PSL_2(8))| = 74$.

Now suppose that $n \geq 3$. By checking the order of $PSL_n(p^m)$ it follows that there is a prime p such that p divides $|PSL_n(p^m)|$ and p^2 does not divide $|PSL_n(p^m)|$ and $p > 100$. So by Lemma 2.2, $|Cent(PSL_n(p^m))| > 100$ unless in the following cases:

$n = 3$ and $p^m = 2, 4, 8, 16, 64, 3, 9, 81, 5, 25, 7, 49, 11, 121, 13, 23, 529, 29, 37$ or
$n = 4$ and $p^m = 2, 4, 8, 3, 9, 5, 7, 11, 13, 23$ or
$n = 5$ and $p^m = 2, 4, 3, 9, 5$ or
$n = 6$ and $p^m = 2, 4, 3, 9, 5$.

By an easy program which is written in The GAP Group (2013) we find the following data:

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$PSL_3(4)$	$\upsilon_7 = 2^6 . 3 . 5$	$PSL_3(49)$	$\upsilon_{19} = 2^9 . 3 . 5^2 . 7^6$
$PSL_3(8)$	$\upsilon_{73} = 2^9 . 3 . 7^2$	$PSL_3(11)$	$\upsilon_{19} = 2^4 . 5^2 . 11^3$
$PSL_3(16)$	$\upsilon_7 = 2^{12} . 3 . 5^2 . 17$	$PSL_3(121)$	$\upsilon_{19} = 2^7 . 3 . 5^2 . 11^6 . 61$
$PSL_3(64)$	$\upsilon_5 = 2^{17} . 3^3 . 7 . 19 . 73$	$PSL_3(13)$	$\upsilon_7 = 2 . 3^2 . 13^3 . 61$
$PSL_3(3)$	$\upsilon_{13} = 2^4 . 3^2$	$PSL_3(23)$	$\upsilon_3 = 7 . 11 . 23^3 . 79$
$PSL_3(9)$	$\upsilon_{13} = 2^7 . 3^5 . 5$	$PSL_3(529)$	$\upsilon_5 = 2^3 . 3^2 . 7 . 11 . 13^2 . 23^6 . 79$
$PSL_3(81)$	$\upsilon_7 = 2^9 . 3^{11} . 5^2 . 41$	$PSL_3(29)$	$\upsilon_3 = 2 . 7 . 13 . 29^3 . 67$
$PSL_3(5)$	$\upsilon_{31} = 2^5 . 5^3$	$PSL_3(37)$	$\upsilon_7 = 2^5 . 3^3 . 19 . 37^3$
$PSL_3(25)$	$\upsilon_{31} = 2^7 . 3 . 5^6 . 13$	$PSL_4(2)$	$\upsilon_5 = 2^4 . 3 . 7$
$PSL_3(7)$	$\upsilon_{19} = 2^5 . 3 . 7^3$	$PSL_4(4)$	$\upsilon_{17} = 2^{10} . 3^4 . 5 . 7$

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$PSL_4(8)$	$\upsilon_{13} = 2^{16} . 3^2 . 7^3 . 73$	$PSL_5(4)$	$\upsilon_7 = 2^{18} . 3 . 5 . 11 . 17 . 31$
$PSL_4(3)$	$\upsilon_{13} = 2^7 . 3^5 . 5$	$PSL_5(3)$	$\upsilon_{13} = 2^5 . 3^8 . 5 . 11^2$
$PSL_4(9)$	$\upsilon_7 = 2^9 . 3^{11} . 5^2 . 41$	$PSL_5(9)$	$\upsilon_7 = 2^8 . 3^{17} . 5 . 11^2 . 41 . 61$
$PSL_4(5)$	$\upsilon_{13} = 2^5 . 3 . 5^6 . 31$	$PSL_5(5)$	$\upsilon_{11} = 2^{11} . 3^2 . 5^9 . 13 . 31$
$PSL_4(7)$	$\upsilon_{19} = 2^9 . 3 . 5^2 . 7^6$	$PSL_6(2)$	$\upsilon_5 = 2^{12} . 3^2 . 7^2 . 31$
$PSL_4(11)$	$\upsilon_7 = 2^7 . 3 . 5^2 . 11^6 . 61$	$PSL_6(4)$	$\upsilon_{11} = 2^{30} . 3^6 . 5^2 . 7^2 . 13 . 17$
$PSL_4(13)$	$\upsilon_5 = 2^5 . 3^4 . 7 . 13^6 . 61$	$PSL_6(3)$	$\upsilon_5 = 2^3 . 3^{14} . 7 . 11^2 . 13^2$
$PSL_4(23)$	$\upsilon_5 = 2^4 . 3 . 7 . 11^3 . 23^6 . 79$	$PSL_6(9)$	$\upsilon_{41} = 2^8 . 3^{28} . 7^2 . 11^2 . 13^2 . 61 . 73$
$PSL_5(2)$	$\upsilon_5 = 2^8 . 3 . 7 . 31$	$PSL_6(5)$	$\upsilon_7 = 2^{12} . 3 . 5^{15} . 11 . 13 . 31 . 71$

By Lemma 2.1 in each case we obtain a contradiction, unless in the case $G \cong PSL_3(2) \cong PSL_2(7)$ and in this case we have $|Cent(G)| = 79$. $\square$

Lemma 3.2. *If L is a classical group or an exceptional group of Lie type, then $|Cent(L)| > 100$, unless in the cases: $L \cong PSL_2(5), PSL_2(7)$ or $PSL_2(8)$.*

Proof. If $L \cong A_n(q) \cong PSL_{n+1}(q)$, then by Lemma 3.1, the only groups satisfying the property $|Cent(L)| \leq 100$ are the groups: $PSL_2(5), PSL_2(7)$ or $PSL_2(8)$.

If $L \cong B_n(q) \cong P\Omega_{2n+1}(q), n > 1$, then by checking the order of L, it follows that there is a prime p such that p divides $|L|$ and p^2 does not divide $|L|$ and $p > 100$.

So by Lemma 2.2, $|Cent(L)| > 100$, unless in the following cases:

$n = 2$ and $q = 2, 4, 8, 32, 3, 9, 27, 5, 7, 11, 13, 17, 23, 31, 41, 43, 47$ or
$n = 3$ and $q = 2, 4, 8, 3, 9, 5, 7, 11,$ or
$n = 4$ and $q = 2, 3,$ or

$n = 5$ and $q = 2, 3$, or
$n = 6$ and $q = 2, 3$.

Since $A_7 \leq P\Omega_5(q)$ in King (2005) and $|Cent(A_7)| = 807$, by Lemma 2.3 we can consider the following cases:

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$P\Omega_7(2)$	$\upsilon_5 = 2^6 . 3^3 . 7$	$P\Omega_7(11)$	$\upsilon_7 = 2^8 . 3^3 . 5^2 . 11^9 . 37 . 61$
$P\Omega_7(4)$	$\upsilon_7 = 2^{17} . 3 . 5^3 . 13 . 17$	$P\Omega_9(2)$	$\upsilon_7 = 2^{14} . 3^3 . 5^2 . 17$
$P\Omega_7(8)$	$\upsilon_5 = 2^{22} . 3^5 . 7^2 . 19 . 73$	$P\Omega_9(3)$	$\upsilon_7 = 2^9 . 3^{14} . 5^2 . 13 . 41$
$P\Omega_7(3)$	$\upsilon_5 = 2^4 . 3^8 . 7 . 13$	$P\Omega_{11}(2)$	$\upsilon_7 = 2^{20} . 3^3 . 5 . 11 . 17 . 31$
$P\Omega_7(9)$	$\upsilon_7 = 2^9 . 3^{17} . 5^3 . 41 . 73$	$P\Omega_{11}(3)$	$\upsilon_7 = 2^8 . 3^{20} . 5 . 11^2 . 13 . 41 . 61$
$P\Omega_7(5)$	$\upsilon_7 = 2^8 . 3 . 5^9 . 13 . 31$	$P\Omega_{13}(2)$	$\upsilon_{11} = 2^{34} . 3^6 . 5^2 . 7^2 . 13 . 17 . 31$
$P\Omega_7(7)$	$\upsilon_{19} = 2^{11} . 3 . 5^2 . 7^9 . 43$	$P\Omega_{13}(3)$	$\upsilon_{41} = 2^{11} . 3^{32} . 5^2 . 7^2 . 11^2 . 13^2 . 61 . 73$

Therefore by Lemma 2.1, in each case we obtain a contradiction. If $L \cong C_n(q) \cong PSp_{2n}(q), n > 2$, then, as before, by checking the order of L, it follows that we must consider the following cases:

$n = 3$ and $q = 2, 4, 8, 3, 9, 5, 7, 11$, or
$n = 4$ and $q = 2, 3$, or
$n = 5$ and $q = 2, 3$, or
$n = 6$ and $q = 2, 3$.

Since $L \cong C_n(q) \cong PSp_{2n}(q)$, we have the following table:

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$PSp_6(2)$	$\upsilon_5 = 2^6 . 3^3 . 7$	$PSp_6(11)$	$\upsilon_7 = 2^8 . 3^3 . 5^2 . 11^9 . 37 . 61$
$PSp_6(4)$	$\upsilon_7 = 2^{17} . 3 . 5^3 . 13 . 17$	$PSp_8(2)$	$\upsilon_7 = 2^{14} . 3^3 . 5^2 . 17$
$PSp_6(8)$	$\upsilon_5 = 2^{22} . 3^5 . 7^2 . 19 . 73$	$PSp_8(3)$	$\upsilon_7 = 2^9 . 3^{14} . 5^2 . 13 . 41$
$PSp_6(3)$	$\upsilon_5 = 2^4 . 3^8 . 7 . 13$	$PSp_{10}(2)$	$\upsilon_7 = 2^{20} . 3^3 . 5 . 11 . 17 . 31$
$PSp_6(9)$	$\upsilon_7 = 2^9 . 3^{17} . 5^3 . 41 . 73$	$PSp_{10}(3)$	$\upsilon_7 = 2^8 . 3^{20} . 5 . 11^2 . 13 . 41 . 61$
$PSp_6(5)$	$\upsilon_7 = 2^8 . 3 . 5^9 . 13 . 31$	$PSp_{12}(2)$	$\upsilon_{11} = 2^{34} . 3^6 . 5^2 . 7^2 . 13 . 17 . 31$
$PSp_6(7)$	$\upsilon_{19} = 2^{11} . 3 . 5^2 . 7^9 . 43$	$PSp_{12}(3)$	$\upsilon_{41} = 2^{11} . 3^{32} . 5^2 . 7^2 . 11^2 . 13^2 . 61 . 73$

So by Lemma 2.1, in each case we obtain a contradiction.

Let $L \cong D_n(q) \cong P\Omega^+_{2n}(q), n > 3$. As before, by checking the order of L, it follows that we must consider the following cases: $D_4(2), D_4(4), D_4(8), D_4(3), D_4(9)$, $D_4(5), D_4(7), D_4(11), D_5(2), D_5(3), D_6(2)$ and $D_6(3)$. Using the fact that $D_n(q) \cong P\Omega^+_{2n}(q)$ and the property $PSL_2(q^t) \leq P\Omega^+_{2^t}(q)$ in Cossidente (2004), only the following cases remain:

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$P\Omega_8^+(2)$	$\upsilon_7 = 2^{11}\,.\,3^4\,.\,5^2$	$P\Omega_{12}^+(2)$	$\upsilon_{11} = 2^{29}\,.\,3^6\,.\,5\,.\,7^2\,.\,17\,.\,31$
$P\Omega_{10}^+(2)$	$\upsilon_7 = 2^{17}\,.\,3^2\,.\,5^2\,.\,17\,.\,31$	$P\Omega_{12}^+(3)$	$\upsilon_{41} = 2^{13}\,.\,3^{28}\,.\,5\,.\,7^2\,.\,11^2\,.\,13^2\,.\,61$
$P\Omega_{10}^+(3)$	$\upsilon_7 = 2^9\,.\,3^{17}\,.\,5\,.\,11^2\,.\,13\,.\,41$		

Therefore by Lemma 2.1, in each case we obtain a contradiction.

If $L \cong G_2(q)$, we obtain a prime p such that p divides $|L|$, and p^2 does not divide $|L|$, and $p > 100$. So by Lemma 2.2, $|Cent(L)| > 100$, unless in the following cases: $G_2(2), G_2(4), G_2(8), G_2(3), G_2(9), G_2(5), G_2(7)$ and $G_2(11)$.

By Yokota (2009), we have $SU_3(q) \leq G_2(q)$. Therefore by Lemmas 2.1 and 2.3 and the following table we obtain a contradiction.

G	$\upsilon_p(G)$
$G_2(2)$	$\upsilon_7 = 2^5\,.\,3^2$
$SU_3(3)$	$\upsilon_7 = 2^5\,.\,3^2$
$SU_3(4)$	$\upsilon_3 = 2^5\,.\,5\,.\,13$
$SU_3(5)$	$\upsilon_7 = 2^4\,.\,3\,.\,5^3$
$SU_3(7)$	$\upsilon_3 = 2^2\,.\,7^3\,.\,43$
$SU_3(8)$	$\upsilon_7 = 2^8\,.\,3^3\,.\,19$
$SU_3(9)$	$\upsilon_{73} = 2^5\,.\,3^5\,.\,5^2$
$SU_3(11)$	$\upsilon_5 = 2\,.\,3^2\,.\,11^3\,.\,37$

If $L \cong F_4(q)$, then we must consider $F_4(2)$ or $F_4(3)$. By Yokota (2009), we have $G_2(q) < F_4(q)$, so we obtain a contradiction.

Let $L \cong E_6(q), E_7(q)$ or $E_8(q)$. As before, by checking the order of L, it follows that we must consider the following case: $E_6(2)$. By Conway et al. (1985), we have $F_4(2) < E_6(2)$ and therefore we obtain a contradiction.

If $L \cong {}^2B_2(q) \cong Sz(q)$, where $q = 2^{2m+1}$, $m \geq 1$, then we must consider ${}^2B_2(8)$ or ${}^2B_2(32)$. Now by Theorem 2.3 we obtain a contradiction.

Let $L \cong {}^2E_6(q)$ or ${}^2F_4(q)$, where $q = 2^{2m+1}$, ${}^3D_4(q)$. we obtain a prime p such that p divides $|L|$, and p^2 does not divide $|L|$, and $p > 100$. So by Lemma 2.2, $|Cent(L)| > 100$, unless in the following cases: ${}^2F_4(2), {}^2E_6(2), {}^2E_6(3), {}^3D_4(2)$ or ${}^3D_4(3)$. By Conway et al. (1985), Wilson (2009) and Lubeck and Malle (1999), we have $PSL_3(3) : 2 < {}^2F_4(2)$, $G_2(q) < {}^3D_4(q)$ and $F_4(q) < {}^2E_6(q)$, respectively.

Thus for all of above groups we have $|Cent(G)| > 100$.

Suppose $L \cong {}^2D_n(q) \cong P\Omega^-_{2n}(q)$, where $n > 3$. We must consider ${}^2D_4(2), {}^2D_4(3)$, ${}^2D_5(2), {}^2D_5(3), {}^2D_6(2), {}^2D_6(3)$ or ${}^2D_7(2)$. Using the fact that ${}^2D_n(q) \cong P\Omega^-_{2n}(q)$ and by The GAP Group (2013), we obtain the following table:

G	$\upsilon_p(G)$
$P\Omega^-_8(2)$	$\upsilon_5 = 2^8 . 3^2 . 7 . 17$
$P\Omega^-_8(3)$	$\upsilon_5 = 2^2 . 3^{10} . 7 . 13 . 41$
$P\Omega^-_{10}(2)$	$\upsilon_7 = 2^{17} . 3^4 . 5 . 11 . 17$
$P\Omega^-_{10}(3)$	$\upsilon_7 = 2^8 . 3^{17} . 5^2 . 13 . 41 . 61$
$P\Omega^-_{12}(2)$	$\upsilon_7 = 2^{23} . 3 . 5^2 . 11 . 13 . 17 . 31$
$P\Omega^-_{12}(3)$	$\upsilon_7 = 2^8 . 3^{23} . 5^2 . 11^2 . 41 . 61 . 73$
$P\Omega^-_{14}(2)$	$\upsilon_{11} = 2^{39} . 3^6 . 5^2 . 7^2 . 13 . 17 . 31 . 43$

So in each case we obtain a contradiction.
If $L \cong {}^2G_2(q)$, where $q = 3^{2m+1}$, then we must consider ${}^2G_2(3)$ or ${}^2G_2(27)$. By The GAP Group (2013), we obtain that $|Cent({}^2G_2(3))| = 548$ and $\upsilon_7({}^2G_2(27)) = 3^8 . 13 . 19 . 37$ and in each case we obtain a contradiction. Finally suppose that $L \cong {}^2A_n(q) \cong U_{n+1}(q) \cong PSU_{n+1}(q)$, then by checking the order of L, it follows that we must consider the following cases: $n = 2$ and $p^m = 2, 4, 8, 3, 9, 27, 5, 7, 11, 17, 19, 23, 31, 37$, or
$n = 3$ and $p^m = 2, 4, 8, 3, 9, 27, 5, 7, 11, 17, 23, 31$ or
$n = 4$ and $p^m = 2, 4, 3$, or
$n = 5$ and $p^m = 2, 4, 3$, or
$n = 6$ and $p^m = 2$, or
$n = 7$ and $p^m = 2$, or
$n = 8$ and $p^m = 2$, or
$n = 9$ and $p^m = 2$.

But using the fact that $A_7 \leq PSU_3(q)$ in King (2005) and $|Cent(A_7)| = 807$, only the following cases remain:

G	$\upsilon_p(G)$	G	$\upsilon_p(G)$
$PSU_4(2)$	$\upsilon_5 = 2^4 . 3^4$	$PSU_5(3)$	$\upsilon_5 = 2^5 . 3^{10} . 7 . 61$
$PSU_4(4)$	$\upsilon_{13} = 2^{12} . 3 . 5^2 . 17$	$PSU_6(2)$	$\upsilon_5 = 2^{12} . 3^5 . 7 . 11$
$PSU_4(8)$	$\upsilon_5 = 2^{16} . 3^7 . 7 . 19$	$PSU_6(4)$	$\upsilon_7 = 2^{29} . 3 . 5^6 . 13 . 17 . 41$
$PSU_4(3)$	$\upsilon_5 = 2^5 . 3^6 . 7$	$PSU_6(3)$	$\upsilon_5 = 2^5 . 3^{14} . 7^2 . 13 . 61$
$PSU_4(5)$	$\upsilon_7 = 2^7 . 3 . 5^6 . 13$	$PSU_7(2)$	$\upsilon_5 = 2^{16} . 3^4 . 7 . 11 . 43$
$PSU_4(7)$	$\upsilon_{43} = 2^9 . 3 . 5^2 . 7^6$	$PSU_8(2)$	$\upsilon_7 = 2^{26} . 3^5 . 5^2 . 11 . 17 . 43$
$PSU_5(2)$	$\upsilon_5 = 2^8 . 3^4 . 11$	$PSU_9(2)$	$\upsilon_7 = 2^{32} . 3^6 . 5^2 . 11 . 17 . 19 . 43$
$PSU_5(4)$	$\upsilon_{13} = 2^{18} . 5^3 . 17 . 41$	$PSU_{10}(2)$	$\upsilon_7 = 2^{38} . 3^6 . 5 . 11^2 . 17 . 19 . 31 . 43$

So in each case we obtain a contradiction. $\square$

Lemma 3.3. *If G is a sporadic simple group, then $|Cent(G)| > 100$.*

Proof. Let S be one of the sporadic groups, M_{11}, M_{12}, M_{22} or J_1. By Conway et al. (1985), $PSL_2(11) < S$. Since $|Cent(PSL_2(11)| = 189$, by Lemma 2.3, we have $|Cent(S)| > 100$.

By Conway et al. (1985),

$$\begin{array}{ll} PSU_3(3) < J_2(p.\ 42), & M_{11} < HS(p.\ 80), \\ PSL_2(19) < J_3(p.\ 82), & M_{11} < McL(p.\ 100), \\ PSL_2(25) < Suz(p.\ 131), & M_{12} < Fi_{22}(p.\ 163), \\ A_{12} < HN(p.\ 166) & \end{array}$$

Thus for all of sporadic groups $S = J_2, HS, J_3, McL, Suz, Fi_{22}$ and HN we have $|Cent(S)| > 100$.
Also by Conway et al. (1985), we have:

$$\begin{array}{ll} M_{22} < M_{23}(p.\ 71), & PSL_2(23) < M_{24}(p.\ 96), \\ HS < Co_3(p.\ 134), & McL < Co_2(p.\ 154), \\ Co_2 < Co_1(p.\ 183), & PSL_2(29) < Ru(p.\ 126), \\ J_1 < O^{\cdot}N(p.\ 132), & G_2(5) < Ly(p.\ 174), \\ S_{12} < Fi_{23}(p.\ 177), & Fi_{23} < Fi_{24}^{'}(p.\ 207), \\ PSL_2(23):2 < J_4(p.\ 190), & PSL_2(19):2 < Th(p.\ 177), \\ Th < B(p.\ 217), & S_3 \times Th < M(p.\ 234), \end{array}$$

Thus for all sporadic groups $S = M_{23}, M_{24}, Co_1, Co_2, Co_3, , Fi_{23}, Fi_{24}^{'}, Ru,$ $O^{\cdot}N, Th, B, J_4, Ly$ and M, we have $|Cent(S)| > 100$.

Finally, by Conway et al. (1985), the Held group He has a subgroup isomorphic to $PSp_4(4)$. So by The GAP Group (2013), we have $\upsilon_{17}(PSp_4(4)) = 2^6 . 3^2 . 5^2$.

Therefore $|Cent(S)| > 100$, for all sporadic simple groups S. The proof is complete. □

The proof of Theorem 1.1:
By Lemmas 3.1, 3.2 and 3.3, we obtain that if G is a finite non-abelian simple group with $|Cent(G)| \leq 100$, then it is isomorphic to one of the following groups $PSL_2(5), PSL_2(7)$ or $PSL_2(8)$.

4. Conclusion

In this paper, we studied all finite non-abelian simple groups G with $|Cent(G)| \leq 100$. We used the fact that every finite non-abelian simple group is isomorphic to one of the following four types of groups: (i) an alternating group A_n for $n \geq 5$, (ii) a classical group, (iii) an exceptional group of Lie type or (iv) one of 26 sporadic simple groups. By counting the number of centralizers of groups in each type, we deduced that a finite non-abelian simple group with $|Cent(G)| \leq 100$ is isomorphic to one of the groups $PSL_2(5), PSL_2(7)$ or $PSL_2(8)$.

References

Abdollahi, A., Jafarian Amiri, S. M., and Mohammadi Hassanabadi, A. (2007). Groups with specific number of centralizers. *Houston Journal of Mathematics*, **33**:43–57.

Ashrafi, A. R. (2000a). Counting the centralizers of some finite groups. *Korean Journal of Computational and Applied Mathematics*, **7**:115–124.

Ashrafi, A. R. (2000b). On finite groups with a given number of centralizers. *Algebra Colloquium*, **7**:139–146.

Ashrafi, A. R. and Taeri, B. (2005). On finite groups with a certain number of centralizers. *Journal of Applied Mathematics and Computing*, **17**:217–227.

Belcastro, S. M. and Sherman, G. J. (1994). Counting centralizers in finite groups. *Mathematics Magazine*, **67**:366–374.

Conway, J. H., Curtis, R. T., Norton, S. P., and Wilson, R. A. (1985). *Atlas of finite groups. Maximal subgroups and ordinary characters for simple groups: With comput. assist. from J. G. Thackray.* Oxford: Clarendon Press.

Cossidente, A. (2004). Maximal subgroups of finite classical groups and their geometry. *Le Matematiche (Catania)*, **59**:173–205.

Foruzanfar, Z. and Mostaghim, Z. (2015). A note on the number of centralizers in finite groups. *Journal of Advanced Research in Pure Mathematics*, **7**:32–40.

Huppert, B. (1967). *Endliche Gruppen I.* Verlag, Berlin.

King, O. H. (2005). The subgroup structure of finite classical groups in terms of geometric configurations. *London Mathematical Society Lecture Note Series*, **327**:29–56.

Lubeck, F. and Malle, G. (1999). (2,3)-generation of exceptional groups. *Journal of the London Mathematical Society*, **59**:109–122.

The GAP Group (2013). *GAP Groups, Algorithms and Programming, Version 4.6.4. (http://www.gap-system.org).*

Wilson, R. A. (2009). *The finite simple groups.* Graduate Texts in Mathematics, Springer-Verlag London, Ltd., London **251**.

Yokota, I. (2009). *Exceptional lie groups.* arxiv:0902.0431v1[math.DG].

Zarrin, M. (2009). On element-centralizers in finite groups. *Archiv der Mathematik*, **93**:497–503.